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# OPERATOR-NORM CONVERGENCE ESTIMATES FOR ELLIPTIC HOMOGENISATION PROBLEMS ON PERIODIC SINGULAR STRUCTURES

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*To the fond memory of Vasily Vasil’evich Zhikov*

## Abstract

For an arbitrary periodic Borel measure  $\mu$ , we prove order  $O(\varepsilon)$  operator-norm resolvent estimates for the solutions to scalar elliptic problems in  $L^2(\mathbb{R}^d, d\mu^\varepsilon)$  with  $\varepsilon$ -periodic coefficients,  $\varepsilon > 0$ . Here  $\mu^\varepsilon$  is the measure obtained by  $\varepsilon$ -scaling of  $\mu$ . Our analysis includes both the case of a measure absolutely continuous with respect to the standard Lebesgue measure and the case of “singular” periodic structures (or “multistructures”), when  $\mu$  is supported by lower-dimensional manifolds.

**Keywords** Homogenisation · Effective properties · Norm-resolvent estimates · Singular structures

## 1 Introduction

The goal of the present work is to prove order-sharp norm-resolvent convergence estimates for partial differential operators with periodic rapidly oscillating coefficients for a wide class of underlying periodic measures. The results on norm-resolvent convergence in homogenisation for the “classical” problem concerning the case of Lebesgue measure go back to the works [5], [6], where the asymptotic analysis of the Green functions of the corresponding problems is carried out, which were followed by the operator-theoretic approach of [2]. An alternative approach, based on the uniform power-series asymptotic analysis of the fibre operators in the associated direct integral, was recently developed in [3]. In the present work we adopt the overall strategy of the latter work, in the setting of an arbitrary periodic Borel measure. As was pointed out in [7], and subsequently discussed in more detail in [8], [9], the analysis of related elliptic problems requires a careful description of the property of (weak) differentiability of functions square integrable with respect to a general Borel measure. In what follows we briefly introduce the tools we employ, namely the Sobolev spaces of quasiperiodic functions with respect to an arbitrary Borel measure (Section 2) and the Floquet transform (Section 3). In Section 4 we formulate and prove our main result (Theorem 4.2). All functions spaces that we use are defined over the field  $\mathbb{C}$  of complex numbers.

Consider a  $Q$ -periodic,  $Q := [0, 1)^d$ , Borel measure  $\mu$ , in  $\mathbb{R}^d$  such that  $\mu(Q) = 1$ , and for each  $\varepsilon > 0$  define an  $\varepsilon$ -periodic measure  $\mu^\varepsilon$  by the formula  $\mu^\varepsilon(B) = \varepsilon^d \mu(\varepsilon^{-1}B)$  for all Borel sets  $B \subset \mathbb{R}^d$ ,  $d \in \mathbb{N}$ . In the present work we study the asymptotic behaviour, as  $\varepsilon \rightarrow 0$ , of the solutions  $u = u^\varepsilon$  to the problems

$$-\nabla \cdot A(\cdot/\varepsilon) \nabla u + u = f, \quad f \in L^2(\mathbb{R}^d, d\mu^\varepsilon), \quad \varepsilon > 0, \quad (1.1)$$

where  $A$  is a positive bounded  $Q$ -periodic  $\mu$ -measurable real-valued matrix function. We aim to prove operators-norm estimates between  $u^\varepsilon$  and the solution to the homogenised equation

$$-\nabla \cdot A^{\text{hom}} \nabla u^0 + u^0 = f, \quad f \in L^2(\mathbb{R}^d, d\mu^\varepsilon), \quad (1.2)$$

with a constant matrix  $A^{\text{hom}}$ , *i.e.* uniform estimates of the form

$$\|u - u^0\|_{L^2(\mathbb{R}^d, d\mu^\varepsilon)} \leq C\varepsilon \|f\|_{L^2(\mathbb{R}^d, d\mu^\varepsilon)},$$

where  $C > 0$  is independent of  $f, \varepsilon$ .

Solutions to (1.1) are understood as a pair  $(u, \nabla u)$  in the space  $H^1(\mathbb{R}^d, d\mu^\varepsilon)$ , defined (*cf.* [9]) as the closure of the set  $\{(\psi, \nabla \psi), \psi \in C_0^\infty(\mathbb{R}^d)\}$  in the norm of  $L^2(\mathbb{R}^d, d\mu^\varepsilon) \oplus [L^2(\mathbb{R}^d, d\mu^\varepsilon)]^d$ . For  $f \in L^2(\mathbb{R}^d, d\mu^\varepsilon)$ , we say that  $(u, \nabla u) \in H^1(\mathbb{R}^d, d\mu^\varepsilon)$  is a solution to (1.1) if

$$\int_{\mathbb{R}^d} A(\cdot/\varepsilon) \nabla u \cdot \overline{\nabla \psi} d\mu^\varepsilon + \int_{\mathbb{R}^d} u \overline{\psi} d\mu^\varepsilon = \int_{\mathbb{R}^d} f \overline{\psi} d\mu^\varepsilon \quad \forall (\psi, \nabla \psi) \in H^1(\mathbb{R}^d, d\mu^\varepsilon). \quad (1.3)$$

Note that for each  $\varepsilon > 0$  the left-hand side of (1.3) is an equivalent inner product on  $H^1(\mathbb{R}^d, d\mu^\varepsilon)$ , and its right-hand side is a linear bounded functional on  $H^1(\mathbb{R}^d, d\mu^\varepsilon)$ . Invoking the Riesz representation theorem (see *e.g.* [1, p. 32]) yields the existence and uniqueness of solution to (1.1).

In what follows we study the resolvent of the operator  $\mathcal{A}^\varepsilon$  with domain

$$\text{dom}(\mathcal{A}^\varepsilon) = \left\{ u \in L^2(\mathbb{R}^d, d\mu^\varepsilon) : \exists \nabla u \in [L^2(\mathbb{R}^d, d\mu^\varepsilon)]^d \text{ such that} \right.$$

$$\left. \int_{\mathbb{R}^d} A(\cdot/\varepsilon) \nabla u \cdot \overline{\nabla \psi} d\mu^\varepsilon + \int_{\mathbb{R}^d} u \overline{\psi} d\mu^\varepsilon = \int_{\mathbb{R}^d} f \overline{\psi} d\mu^\varepsilon \quad \forall (\psi, \nabla \psi) \in H^1(\mathbb{R}^d, d\mu^\varepsilon) \text{ for some } f \in L^2(\mathbb{R}^d, d\mu^\varepsilon) \right\}. \quad (1.4)$$

defined by the formula  $\mathcal{A}^\varepsilon u = f - u$  whenever  $f \in L^2(\mathbb{R}^d, d\mu^\varepsilon)$  and  $u \in \text{dom}(\mathcal{A}^\varepsilon)$  are related as in (1.4). Note that while in general for a given  $u \in L^2(\mathbb{R}^d, d\mu^\varepsilon)$  there may be more than one element  $(u, \nabla u) \in H^1(\mathbb{R}^d, d\mu^\varepsilon)$ , the uniqueness of solution to (1.1) implies that for each function  $u \in \text{dom}(\mathcal{A}^\varepsilon)$  there is exactly one gradient  $\nabla u$  such that the identity in (1.4) holds.

Clearly, the operator  $\mathcal{A}^\varepsilon$  is symmetric. By an argument similar to [7, Section 7.1], we infer that  $\text{dom}(\mathcal{A}^\varepsilon)$  is dense in  $L^2(\mathbb{R}^d, d\mu^\varepsilon)$ . Indeed, it follows from (1.4) that if  $f \in L^2(\mathbb{R}^d, d\mu^\varepsilon)$ , and  $u, v \in \text{dom}(\mathcal{A}^\varepsilon)$  are such that  $\mathcal{A}^\varepsilon u + u = f$ ,  $\mathcal{A}^\varepsilon v + v = u$ , then

$$\int_{\mathbb{R}^d} f \overline{v} = \int_{\mathbb{R}^d} |u|^2. \quad (1.5)$$

The identity (1.5) implies that if  $f$  is orthogonal to  $\text{dom}(\mathcal{A}^\varepsilon)$  then  $u = 0$ , and hence  $f = 0$ . Furthermore,  $\mathcal{A}^\varepsilon$  is self-adjoint. Indeed, suppose that  $w \in \text{dom}((\mathcal{A}^\varepsilon)^*) \subset L^2(\mathbb{R}^d, d\mu^\varepsilon)$ , so for some  $g \in L^2(\mathbb{R}^d, d\mu^\varepsilon)$  one has

$$\int_{\mathbb{R}^d} (\mathcal{A}^\varepsilon u) \overline{w} d\mu^\varepsilon = \int_{\mathbb{R}^d} u \overline{g} d\mu^\varepsilon \quad \forall u \in \text{dom}(\mathcal{A}^\varepsilon).$$

Consider the solution  $v$  to the problem

$$\mathcal{A}^\varepsilon v + v = g + w.$$

Then for all  $u \in \text{dom}(\mathcal{A}^\varepsilon)$  one has

$$\int_{\mathbb{R}^d} (\mathcal{A}^\varepsilon u + u) \overline{w} = \int_{\mathbb{R}^d} u \overline{(g + w)} = \int_{\mathbb{R}^d} u \overline{(\mathcal{A}^\varepsilon v + v)} = \int_{\mathbb{R}^d} (\mathcal{A}^\varepsilon u + u) \overline{v},$$

where we use the fact that  $\mathcal{A}^\varepsilon$  is symmetric and  $u, v \in \text{dom}(\mathcal{A}^\varepsilon)$ . Since  $\mathcal{A}^\varepsilon u + u$  is an arbitrary element of  $L^2(\mathbb{R}^d, d\mu^\varepsilon)$ , it follows that  $w = v$ , and in particular  $w \in \text{dom}(\mathcal{A}^\varepsilon)$ .

Similarly, we define the operator  $\mathcal{A}^{\text{hom}}$  associated with the problem (1.2), so that (1.2) holds if and only if  $u^0 = (\mathcal{A}^{\text{hom}} + I)^{-1}f$ .

All gradients, integrals and differential operators below, unless indicated explicitly otherwise, are understood appropriately with respect to the measure  $\mu$ . Whenever we write  $\int_Q$ , we imply integration with respect to the measure  $\mu$  and interchangeably use the notation and  $L^2(Q, d\mu)$  and  $L^2(Q)$  for the Lebesgue space of functions that are square integrable on  $Q$  with respect to  $\mu$ . Throughout the paper we use the notation  $e_\varkappa$  for the exponent  $\exp(i\varkappa \cdot y)$ ,  $y \in Q$ ,  $\varkappa \in [-\pi, \pi]^d$ , and a similar notation  $e_\theta$  for the exponent  $\exp(i\theta \cdot x)$ ,  $x \in \mathbb{R}^d$ ,  $\theta \in \varepsilon^{-1}[-\pi, \pi]^d$ . We denote by  $C_\#^\infty$  the set of  $Q$ -periodic functions in  $C^\infty(\mathbb{R}^d)$ , and  $\partial_j \varphi$ ,  $\nabla \varphi$ ,  $\nabla(e_\varkappa \varphi)$   $\nabla(e_\varepsilon \theta \varphi)$  stand for the classical derivatives and gradients of smooth functions  $\varphi$ ,  $e_\varkappa \varphi$ ,  $e_\varepsilon \theta \varphi$ . The symbol “:=” stands is used to denote the expression on the right-hand side of the symbol by its left-hand side.

## 2 Sobolev spaces of quasiperiodic functions

The material of this section applies to an arbitrary Borel measure  $\mu$  on  $Q$ . The following definition is motivated by [7, Section 3.1], [8].

**Definition 1.** For each  $\varkappa \in [-\pi, \pi]^d := Q'$  we define the space  $H_\varkappa^1$  as the closure, with respect to the natural norm of the direct sum  $L^2(Q) \oplus [L^2(Q)]^d$ , of the set  $\{(e_\varkappa \varphi, \nabla(e_\varkappa \varphi)) : \varphi \in C_\#^\infty\}$ . We use the notation  $H_\#^1$  for the space  $H_\varkappa^1$ ,  $\varkappa = 0$ . For  $(u, v) \in H_\varkappa^1$  we keep the usual notation  $\nabla u$  for the second element  $v$  in the pair.

As discussed in [7], [8], [9], there may be different elements in  $H_\varkappa^1$  whose first components coincide. Indeed, for any  $(u, v) \in H_\varkappa^1$  and a vector function  $w$  obtained as the limit in  $[L^2(Q)]^d$  of the classical gradients  $\nabla(e_\varkappa \varphi_n)$  for a sequence  $\varphi_n \in C_\#^\infty$  converging to zero in  $L^2(Q)$  (“gradient of zero”), the pair  $(u, v + w)$  is also an element of  $H_\varkappa^1$ . Furthermore, there is a natural one-to-one mapping between  $H_\varkappa^1$  and  $H_\#^1$ : for any element  $(u, v) \in H_\varkappa^1$  the pair  $(\overline{e_\varkappa} u, \overline{e_\varkappa}(v - iu\varkappa))$  is an element of  $H_\#^1$  and for all  $(\tilde{u}, \tilde{v}) \in H_\#^1$  one has  $\tilde{v} = \overline{e_\varkappa}(v - iu\varkappa)$  for some  $(u, v) \in H_\varkappa^1$ . In view of this, for  $(\tilde{u}, \tilde{v}) \in H_\#^1$  we often write  $\tilde{v} = \nabla \tilde{u} = \overline{e_\varkappa} \nabla(e_\varkappa \tilde{u}) - i\tilde{u}\varkappa$ , where either  $\nabla \tilde{u}$  or  $\nabla(e_\varkappa \tilde{u})$  is defined up to a gradient of zero.

Suppose that  $A \in [L^\infty(Q, d\mu)]^{d \times d}$  is a pointwise positive and symmetric real-valued matrix function such that  $A^{-1} \in [L^\infty(Q, d\mu)]^{d \times d}$ , and for each  $\varkappa \in Q'$  consider the operator  $\mathcal{A}_\varkappa$  with domain (cf. (1.4))

$$\text{dom}(\mathcal{A}_\varkappa) = \left\{ u \in L^2(Q) : \exists \nabla(e_\varkappa u) \in [L^2(Q)]^d \text{ such that} \right. \\ \left. \int_Q A \nabla(e_\varkappa u) \cdot \overline{\nabla(e_\varkappa \varphi)} + \int_Q u \overline{\varphi} = \int_Q F \overline{\varphi} \quad \forall \varphi \in C_\#^\infty \quad \text{for some } F \in L^2(Q) \right\}, \quad (2.6)$$

defined by the formula  $\mathcal{A}_\varkappa u = F - u$  whenever  $F \in L^2(Q)$  and  $u \in \text{dom}(\mathcal{A}_\varkappa)$  are related as described in the definition of  $\text{dom}(\mathcal{A}_\varkappa)$ . Notice that by the definition of  $H_\varkappa^1$ , the set  $C_\#^\infty$  of test functions in the identity in (2.6) can be equivalently replaced by  $H_\varkappa^1$ . As discussed in the previous section for the case of operator  $\mathcal{A}^\varepsilon$ , since for  $F = 0$  one has  $u = 0$ ,  $\nabla(e_\varkappa u) = 0$ , there is exactly one gradient  $\nabla(e_\varkappa u)$  for which (2.6) holds. Also, by an argument similar to the case of  $\mathcal{A}^\varepsilon$ , the domain  $\text{dom}(\mathcal{A}_\varkappa)$  is dense in  $L^2(Q)$  and  $\mathcal{A}_\varkappa$  is self-adjoint.

In what follows, we identify  $H_\#^1$  and the the set of the first components of its elements, bearing in mind that the gradient of a function in  $H_\#^1$  may not be unique. We also denote by  $H_{\#,0}^1$  the (closed) subspace of  $H_\#^1$  consisting of functions with zero  $\mu$ -mean over  $Q$ .

## 3 Floquet transform

In this section we define a representation for functions in  $L^2(\mathbb{R}^d, d\mu)$  unitarily equivalent to “Gelfand transform”, introduced in [4] for the case of the Lebesgue measure. The properties of the Gelfand transform with

respect to the measure  $\mu$  are discussed in detail in [9], and here we give the definition of its Floquet version as well as the key property concerning the equation (1.1). We first define a “scaled” version of the Floquet transform (cf. [3]).

**Definition 2.** For  $\varepsilon > 0$  and  $u \in C_0^\infty(\mathbb{R}^d)$ , the  $\varepsilon$ -Floquet transform  $\mathcal{F}_\varepsilon u$  is the function

$$(\mathcal{F}_\varepsilon u)(z, \theta) = \left( \frac{\varepsilon}{2\pi} \right)^{d/2} \sum_{n \in \mathbb{Z}^d} u(z + \varepsilon n) \exp(-i\varepsilon n \cdot \theta), \quad z \in \varepsilon Q, \quad \theta \in \varepsilon^{-1} Q'.$$

The mapping  $\mathcal{F}_\varepsilon$  preserves the norm, in the sense that

$$\|\mathcal{F}_\varepsilon u\|_{L^2(\varepsilon^{-1} Q' \times \varepsilon Q, d\theta \times d\mu^\varepsilon)} = \|u\|_{L^2(\mathbb{R}^d, d\mu^\varepsilon)},$$

and it can therefore be extended to an isometry  $\mathcal{F}_\varepsilon : L^2(\mathbb{R}^d, d\mu^\varepsilon) \mapsto L^2(\varepsilon^{-1} Q' \times \varepsilon Q, d\theta \times d\mu^\varepsilon)$ , for which we use the same term  $\varepsilon$ -Floquet transform. Note that the inverse of  $\mathcal{F}_\varepsilon$  is given by

$$(\mathcal{U}_\varepsilon g)(z) = \left( \frac{\varepsilon}{2\pi} \right)^{d/2} \int_{\varepsilon^{-1} Q'} g(\theta, z) d\theta, \quad z \in \mathbb{R}^d, \quad g \in L^2(\varepsilon^{-1} Q' \times \varepsilon Q, d\theta \times d\mu^\varepsilon), \quad (3.7)$$

where for each  $\theta \in \varepsilon^{-1} Q'$  the function  $g \in L^2(\varepsilon^{-1} Q' \times \varepsilon Q, d\theta \times d\mu^\varepsilon)$  is extended as  $\theta$ -quasiperiodic function to the whole of  $\mathbb{R}^d$  so that

$$g(\theta, z) = \tilde{g}(\theta, z) \exp(i\varepsilon z \cdot \theta), \quad z \in \mathbb{R}^d, \quad \tilde{g}(\theta, \cdot) \text{ } \varepsilon Q\text{-periodic}.$$

Indeed, for all such functions  $g$  the right-hand side (3.7) is well defined and returns a function in  $L^2(\mathbb{R}^d)$ , cf. [9]:

$$\|\mathcal{U}_\varepsilon g\|_{L^2(\mathbb{R}^d)}^2 = \sum_{n \in \mathbb{Z}^d} \|(\mathcal{U}_\varepsilon g)(\cdot + \varepsilon n)\|_{L^2(\varepsilon Q)}^2 = \sum_{n \in \mathbb{Z}^d} \|\hat{g}_n\|_{L^2(\varepsilon Q)}^2 = \int_{\varepsilon Q} \int_{\varepsilon^{-1} Q'} |(\mathcal{U}_\varepsilon g)(\cdot, \theta)|^2 d\theta d\mu^\varepsilon,$$

where, for each  $z \in \varepsilon Q$ ,

$$\hat{g}_n(z) := \left( \frac{\varepsilon}{2\pi} \right)^{d/2} \int_{\varepsilon^{-1} Q'} g(\theta, z) \exp(i\varepsilon n \cdot \theta) d\theta, \quad n \in \mathbb{Z}^d,$$

are the Fourier coefficients of the  $\varepsilon^{-1} Q'$ -periodic function  $g(\cdot, z)$ . Since the image of  $\mathcal{U}_\varepsilon$  contains  $C_0^\infty(\mathbb{R}^d)$  and for all  $u \in C_0^\infty(\mathbb{R}^d)$  one has  $u = \mathcal{U}_\varepsilon \mathcal{F}_\varepsilon u$ , it follows that  $\mathcal{F}_\varepsilon$  is one-to-one and thus, unitary.

Combining the  $\varepsilon$ -Floquet transform and the unitary scaling transform

$$\begin{aligned} \mathcal{T}_\varepsilon h(\theta, y) &:= \varepsilon^{d/2} h(\theta, \varepsilon y), \quad \theta \in \varepsilon^{-1} Q', \quad y \in Q, \quad \forall h \in L^2(\varepsilon^{-1} Q' \times \varepsilon Q, d\theta \times d\mu^\varepsilon), \\ (\mathcal{T}_\varepsilon^{-1} h)(\theta, z) &= \varepsilon^{-d/2} h(\theta, z/\varepsilon), \quad \theta \in \varepsilon^{-1} Q', \quad z \in \varepsilon Q, \quad \forall h \in L^2(\varepsilon^{-1} Q' \times Q, d\theta \times d\mu), \end{aligned}$$

we obtain a representation for the operator  $\mathcal{A}^\varepsilon$ , as follows.

**Proposition 3.1.** For each  $\varepsilon > 0$  the operator  $\mathcal{A}^\varepsilon$  is unitarily equivalent to the direct integral of the family  $\mathcal{A}_{\varepsilon\theta}$ ,  $\theta \in \varepsilon^{-1} Q'$ , namely

$$(\mathcal{A}^\varepsilon + I)^{-1} = \mathcal{F}_\varepsilon^{-1} \mathcal{T}_\varepsilon^{-1} \int_{\varepsilon^{-1} Q'}^\oplus e_{\varepsilon\theta} (\varepsilon^{-2} \mathcal{A}_{\varepsilon\theta} + I)^{-1} \overline{e_{\varepsilon\theta}} d\theta \mathcal{T}_\varepsilon \mathcal{F}_\varepsilon,$$

where  $\overline{e_{\varepsilon\theta}}$ ,  $e_{\varepsilon\theta}$  represent the operators of multiplication by  $\overline{e_{\varepsilon\theta}}$ ,  $e_{\varepsilon\theta}$ .

*Sketch of proof.* The argument is similar to [9]. Taking first solutions  $(u, \nabla u) \in H^1(\mathbb{R}^d, d\mu^\varepsilon)$  to (1.1) with  $f \in C_0^\infty(\mathbb{R}^d)$ , whose both components can be shown to decay exponentially at infinity, cf. [9, Proposition 5.3], we denote, for each such  $u$ , the “periodic amplitude” of its scaled  $\varepsilon$ -Floquet transform:

$$u_\theta^\varepsilon := \overline{e_{\varepsilon\theta}} \mathcal{T}_\varepsilon \mathcal{F}_\varepsilon u = \left( \frac{\varepsilon^2}{2\pi} \right)^{d/2} \sum_{n \in \mathbb{Z}^d} u(\varepsilon y + \varepsilon n) \exp(-i(\varepsilon y + \varepsilon n) \cdot \theta). \quad (3.8)$$

Note that for any choice of the gradient  $\nabla u$  (and hence for the one entering (1.1)) the expression

$$\nabla(e_{\varepsilon\theta} u_\theta^\varepsilon)(y) = \varepsilon \left( \frac{\varepsilon^2}{2\pi} \right)^{d/2} \sum_{n \in \mathbb{Z}^d} \nabla u(\varepsilon y + \varepsilon n) \exp(-i(\varepsilon y + \varepsilon n) \cdot \theta), \quad y \in Q,$$

is a gradient of  $e_{\varepsilon\theta} u_\theta^\varepsilon$ , in the sense that  $(e_{\varepsilon\theta} u_\theta^\varepsilon, \nabla(e_{\varepsilon\theta} u_\theta^\varepsilon)) \in H_{\varepsilon\theta}^1$ , as shown by considering an appropriate sequence  $\varphi_n \in C_0^\infty(\mathbb{R}^d)$  whose classical gradients converge to  $\nabla u$  in  $L^2(\mathbb{R}^d, d\mu^\varepsilon)$ . Therefore

$$\varepsilon^{-2} \int_Q A \nabla(e_{\varepsilon\theta} u_\theta^\varepsilon) \cdot \overline{\nabla(e_{\varepsilon\theta} \varphi)} d\mu + \int_Q e_{\varepsilon\theta} u_\theta^\varepsilon \overline{e_{\varepsilon\theta} \varphi} d\mu = \int_Q e_{\varepsilon\theta} F \overline{e_{\varepsilon\theta} \varphi} d\mu \quad \forall \varphi \in C_\#^\infty, \quad (3.9)$$

where  $F = \overline{e_{\varepsilon\theta}} \mathcal{T}_\varepsilon \mathcal{F}_\varepsilon f$ . The density of  $f \in C_0^\infty(\mathbb{R}^d)$  in  $L^2(\mathbb{R}^d, d\mu^\varepsilon)$  implies the claim.  $\square$

In what follows we study the asymptotic behaviour of the solutions  $u_\theta^\varepsilon$  to the problems

$$\varepsilon^{-2} \overline{e_{\varepsilon\theta}} \nabla \cdot A \nabla(e_{\varepsilon\theta} u_\theta^\varepsilon) + u_\theta^\varepsilon = F, \quad \varepsilon > 0, \quad \theta \in \varepsilon^{-1} Q', \quad (3.10)$$

understood in the sense of the identity (3.9).

## 4 Asymptotic approximation of $u_\theta^\varepsilon$

Henceforth we assume that the measure  $\mu$  is ergodic, i.e. whenever  $\varphi_n \in C_\#^\infty$  and the classical gradients  $\nabla \varphi_n$  converge to zero in  $[L^2(Q, d\mu)]^d$ , there exists a constant  $c$  such that  $\varphi_n \rightarrow c$  in  $L^2(Q, d\mu)$ . We also assume (cf. [9]) that the embedding  $H_\#^1(Q, d\mu) \subset L^2(Q, d\mu)$  is compact. In what follows we assume that  $A$  is a scalar matrix. The analysis of the general case is similar: the modifications required concern the condition on the mean of the unit-cell solutions defined next.

Consider the vector  $N = (N_1, N_2, \dots, N_d)$  of solutions to the unit cell problems<sup>1</sup>

$$-\nabla \cdot A \nabla N_j = \partial_j A, \quad \int_Q A N_j = 0, \quad j = 1, 2, \dots, d. \quad (4.11)$$

The right-hand side of (4.11) is understood as an element of the space  $(H_\#^1)^*$  of linear continuous functionals on  $H_\#^1$ : for a test function  $\varphi \in C_\#^\infty$  the action of  $\partial_j A$  on  $\varphi$  is given by

$$\langle \partial_j A, \varphi \rangle = \int_Q A \overline{\partial_j \varphi},$$

and the action of the same functional on the whole space  $H_\#^1$  is obtained by closure. In particular, for a pair  $\mathcal{V} = (v, \nabla v) \in H_\#^1$  we have

$$\langle \partial_j A, \mathcal{V} \rangle = \int_Q A \overline{\partial_j v}. \quad (4.12)$$

**Proposition 4.1.** *For each  $j = 1, 2, \dots, d$ , there exists a unique solution  $N_j \in H_\#^1$  to (4.11).*

<sup>1</sup>In the case of matrix-valued  $A$ , the condition on the mean of the solutions  $N_j$ ,  $j = 1, 2, \dots, d$ , is replaced by  $\int_Q (A\theta \cdot \theta) N_j = 0$  for  $\theta \neq 0$ , with no condition imposed for  $\theta = 0$ , so the mean of  $N_j$  (but not its gradient) depends on  $\theta$ .

*Proof.* It follows from the above assumptions on the measure  $\mu$  that the Poincaré inequality holds:<sup>2</sup>

$$\left\| u - \int_Q u \right\|_{L^2(Q)} \leq C \|\nabla u\|_{[L^2(Q)]^d}, \quad C > 0, \quad \forall (u, \nabla u) \in H_{\#}^1. \quad (4.13)$$

Therefore, the sesquilinear form

$$\int_Q A \nabla u \cdot \overline{\nabla v}, \quad (u, \nabla u), (v, \nabla v) \in H_{\#}^1,$$

is bounded and coercive, and hence defines an equivalent inner product in  $H_{\#}^1$ . Bearing in mind that (4.12) is a linear bounded functional on  $H_{\#}^1$ , we infer by the Riesz representation theorem (see *e.g.* [1, p. 32]) that for each  $j = 1, 2, \dots, d$ , the equation

$$-\nabla \cdot A \nabla u = \partial_j A,$$

has a unique solution in  $\tilde{N}_j \in H_{\#}^1$ , and therefore its arbitrary solution in  $H_{\#}^1$  has the form  $\tilde{N}_j + a$ ,  $a \in \mathbb{C}$ . Setting

$$a = - \left( \int_Q A \right)^{-1} \int_Q A \tilde{N}_j, \quad N_j := \tilde{N}_j + a,$$

concludes the proof.  $\square$

**Theorem 4.2.** *Suppose that for the measure  $\mu$  there exists  $C_P = C_P(\mu) > 0$  such that for all  $\varkappa \in Q'$  and  $(e_{\varkappa} u, \nabla(e_{\varkappa} u)) \in H_{\varkappa}^1$  the Poincaré-type inequality (cf. (4.13))*

$$\left\| u - \int_Q u \right\|_{L^2(Q)} \leq C_P \|\nabla(e_{\varkappa} u)\|_{[L^2(Q)]^d} \quad (4.14)$$

*holds. Then the following estimate holds for the solutions to (3.10) with a constant  $C > 0$  independent of  $\varepsilon$ ,  $\theta$ ,  $F$ :*

$$\|u_{\theta}^{\varepsilon} - c_{\theta}\|_{L^2(Q)} \leq C \varepsilon \|F\|_{L^2(Q)}, \quad (4.15)$$

where

$$c_{\theta} = c_{\theta}(F) := \left( \theta \cdot \left\{ \int_Q A(\nabla N + I) \right\} \theta + 1 \right)^{-1} \int_Q F, \quad \theta \in \varepsilon^{-1} Q'. \quad (4.16)$$

**Corollary 4.3.** *Under the conditions of the above theorem, there exists  $C > 0$  such that*

$$\|u^{\varepsilon} - u^0\|_{L^2(\mathbb{R}^d, d\mu^{\varepsilon})} \leq C \varepsilon \|f\|_{L^2(\mathbb{R}^d, d\mu^{\varepsilon})} \quad \forall \varepsilon > 0, \quad f \in L^2(\mathbb{R}^d, d\mu^{\varepsilon}),$$

where  $u^{\varepsilon}$  are the solutions to the original family (1.1) and  $u^0$  is the solution to the homogenised equation (1.2) with

$$A^{\text{hom}} := \int_Q A(\nabla N + I).$$

*Proof of Corollary 4.3.* Consider  $f \in L^2(\mathbb{R}^d, d\mu^{\varepsilon})$  and denote (see (3.8))  $f_{\theta}^{\varepsilon} := \overline{e_{\varepsilon\theta}} \mathcal{T}_{\varepsilon} \mathcal{F}_{\varepsilon} f$ , so that

$$\int_Q f_{\theta}^{\varepsilon} = \widehat{f}(\theta), \quad \theta \in \varepsilon^{-1} Q', \quad \text{where} \quad \widehat{f}(\theta) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(z) \overline{e_{\theta}} d\mu(z), \quad \theta \in \mathbb{R}^d.$$

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<sup>2</sup>The inequality (4.13) follows from the fact that, under our assumptions on the measure  $\mu$ , the spectrum of the Laplacian on the torus is discrete and its eigenvalue zero is simple.

Also, consider the solutions  $u_\theta^\varepsilon$  to (3.10) with  $F = f_\theta^\varepsilon$ . Using Proposition 3.1 we obtain

$$\begin{aligned} (\mathcal{A}^\varepsilon + I)^{-1}f - (\mathcal{A}^{\text{hom}} + I)^{-1}f &= \mathcal{F}_\varepsilon^{-1}\mathcal{T}_\varepsilon^{-1}e_{\varepsilon\theta}(\varepsilon^{-2}\mathcal{A}_{\varepsilon\theta} + I)^{-1}f_\theta^\varepsilon - (\mathcal{A}^{\text{hom}} + I)^{-1}f \\ &= \mathcal{F}_\varepsilon^{-1}\mathcal{T}_\varepsilon^{-1}e_{\varepsilon\theta}u_\theta^\varepsilon - (\mathcal{A}^{\text{hom}} + I)^{-1}f \\ &= \{\mathcal{F}_\varepsilon^{-1}\mathcal{T}_\varepsilon^{-1}e_{\varepsilon\theta}u_\theta^\varepsilon - \mathcal{F}_\varepsilon^{-1}\mathcal{T}_\varepsilon^{-1}e_{\varepsilon\theta}c_\theta(f_\theta^\varepsilon)\} + \{\mathcal{F}_\varepsilon^{-1}\mathcal{T}_\varepsilon^{-1}e_{\varepsilon\theta}c_\theta(f_\theta^\varepsilon) - (\mathcal{A}^{\text{hom}} + I)^{-1}f\}, \end{aligned}$$

where the operators  $\mathcal{A}^\varepsilon$ ,  $\mathcal{A}^{\text{hom}}$  are defined at the end of Section 1. In view of Theorem 4.2, the unitary property of  $\mathcal{F}_\varepsilon$ ,  $\mathcal{T}_\varepsilon$  and the operator of multiplication by  $e_{\varepsilon\theta}$ , as well as the fact that

$$\begin{aligned} &\mathcal{F}_\varepsilon^{-1}\mathcal{T}_\varepsilon^{-1}e_{\varepsilon\theta}(\theta \cdot A^{\text{hom}}\theta + 1)^{-1}\widehat{f}(\theta) - (2\pi)^{-d/2} \int_{\mathbb{R}^d} (\theta \cdot A^{\text{hom}}\theta + 1)^{-1}\widehat{f}(\theta)e_\theta d\theta \\ &= (2\pi)^{-d/2} \left\{ \int_{\varepsilon^{-1}Q'} (\theta \cdot A^{\text{hom}}\theta + 1)^{-1}\widehat{f}(\theta)e_\theta d\theta - \int_{\mathbb{R}^d} (\theta \cdot A^{\text{hom}}\theta + 1)^{-1}\widehat{f}(\theta)e_\theta d\theta \right\} \\ &= \int_{\mathbb{R}^d \setminus \varepsilon^{-1}Q'} (\theta \cdot A^{\text{hom}}\theta + 1)^{-1}\widehat{f}(\theta)e_\theta d\theta, \end{aligned}$$

we obtain

$$\|(\mathcal{A}^\varepsilon + I)^{-1}f - (\mathcal{A}^{\text{hom}} + I)^{-1}f\|_{L^2(\mathbb{R}^d, d\mu^\varepsilon)} \leq C\varepsilon\|f\|_{L^2(\mathbb{R}^d, d\mu^\varepsilon)} + \frac{\varepsilon^2}{\|A^{\text{hom}}\|^2 + \varepsilon^2}\|\widehat{f}\|_{L^2(\mathbb{R}^d)},$$

from which the claim follows.  $\square$

We now proceed to the proof of Theorem 4.2. Motivated by formal asymptotics in powers of  $\varepsilon$ , we consider the function

$$U_\theta^\varepsilon := c_\theta + i\varepsilon N_j \theta_j c_\theta + \varepsilon^2 R_\theta^\varepsilon, \quad (4.17)$$

where  $\nabla N_j$ ,  $j = 1, 2, \dots, d$ , are defined by (4.11), and the “remainder”  $R_\theta^\varepsilon \in H_\#^1$  solves

$$\begin{aligned} &-\overline{e_{\varepsilon\theta}}\nabla \cdot A\nabla(e_{\varepsilon\theta}R_\theta^\varepsilon) + \varepsilon^2 \int_Q R_\theta^\varepsilon = F + \varepsilon^{-2}\overline{e_{\varepsilon\theta}}\nabla \cdot A\nabla(e_{\varepsilon\theta}c_\theta) + i\varepsilon^{-1}\overline{e_{\varepsilon\theta}}\nabla \cdot A\nabla(e_{\varepsilon\theta}N_j\theta_j)c_\theta - c_\theta \\ &\equiv F + i\nabla \cdot (iN_j\theta_j A\theta)c_\theta + i\theta \cdot A\nabla(iN_j\theta_j)c_\theta - i\varepsilon N_j\theta_j\theta \cdot A\theta c_\theta - \theta \cdot A\theta c_\theta - c_\theta =: H_\theta^\varepsilon, \end{aligned} \quad (4.18)$$

where  $H_\theta^\varepsilon$  is treated as an element of the space  $(H_\#^1)^*$ . Here for all  $\varkappa \in Q'$  we set

$$\nabla e_\varkappa = i e_\varkappa \varkappa, \quad \nabla(e_\varkappa N_j) = e_\varkappa(iN_j \varkappa + \nabla N_j), \quad 1, 2, \dots, d. \quad (4.19)$$

The second equality in (4.18) is verified by taking  $\varphi \in C_\#^\infty$ , noticing that

$$\begin{aligned} &\left\langle \varepsilon^{-2}\overline{e_{\varepsilon\theta}}\nabla \cdot A\nabla e_{\varepsilon\theta} + i\varepsilon^{-1}\overline{e_{\varepsilon\theta}}\nabla \cdot A\nabla(e_{\varepsilon\theta}N_j\theta_j), \varphi \right\rangle \\ &= - \int_Q \left( \varepsilon^{-2}Aie_{\varepsilon\theta}\varepsilon\theta \cdot \overline{\nabla(e_{\varepsilon\theta}\varphi)} + i\varepsilon^{-1}Ae_{\varepsilon\theta}\theta_j(iN_j\varepsilon\theta + \nabla N_j) \cdot \overline{\nabla(e_{\varepsilon\theta}\varphi)} \right) \\ &= - \int_Q \left( \varepsilon^{-2}Aie_{\varepsilon\theta}\varepsilon\theta \cdot \overline{(ie_{\varepsilon\theta}\varphi\varepsilon\theta + e_{\varepsilon\theta}\nabla\varphi)} + i\varepsilon^{-1}Ae_{\varepsilon\theta}\theta_j(iN_j\varepsilon\theta + \nabla N_j) \cdot \overline{(ie_{\varepsilon\theta}\varphi\varepsilon\theta + e_{\varepsilon\theta}\nabla\varphi)} \right), \end{aligned}$$

and finally using (4.11). Note that for  $c_\theta$  defined by (4.16), the condition  $\langle H_\theta^\varepsilon, 1 \rangle = 0$  holds, and in the case  $\theta = 0$  the average over  $Q$  of the solution  $R_\theta^\varepsilon$  to (4.18) vanishes.

The estimate (4.14) and the classical Riesz representation theorem [1, p.32] imply that for each  $\varepsilon > 0$ ,  $\theta \in \varepsilon^{-1}Q'$ , there exists a unique solution  $R_\theta^\varepsilon \in H_\#^1$  to the problem (4.18).



## 5 Discussion of the validity of (4.14) for some singular measures

Note that that for all  $(e_{\varkappa}u, \nabla(e_{\varkappa}u)) \in H_{\varkappa}^1$  one has  $\nabla(e_{\varkappa}u) = e_{\varkappa}(iu\varkappa + \nabla u)$  for some  $(u, \nabla u) \in H_{\#}^1$ . Therefore, in order to prove (4.14) it suffices to minimise

$$\left( \int_Q |u|^2 d\mu \right)^{-1} \int_Q |iu\varkappa + \nabla u|^2 d\mu, \quad u \in H_{\#}^1, \quad \int_Q u = 0, \quad u \neq 0,$$

and then take the infimum over  $\varkappa \in [-\pi, \pi)^d$ .

For the case when  $\mu$  is the Lebesgue measure, one has  $(C_P(\mu))^{-2} = \min_{\varkappa} (\varkappa + 2\pi)^2 = \pi^2$ . The same value of  $C_P$  applies to the case of the linear measure supported by the “square grid”, see [9, Section 9.2].

Consider a finite set  $\{\mathcal{H}_j\}_j$  of hyperplanes of dimension  $d$  or smaller,  $\mathcal{H}_j$  is parallel to the coordinate axis for all  $j$  and  $\mathcal{H}_j$  is not a subset of  $\mathcal{H}_k$  for all  $j, k$ . Define the measure  $\mu$  on  $Q$  by the formula

$$\mu(B) = \left( \sum_j |\mathcal{H}_j \cap Q|_j \right)^{-1} \sum_j |\mathcal{H}_j \cap B|_j \quad \text{for all Borel } B \subset Q,$$

where  $|\cdot|_j$  represents the  $d_j$ -dimensional Lebesgue measure,  $d_j = \dim(\mathcal{H}_j)$ . Then the assumptions of Theorem 4.2 hold with  $C_P(\mu) = 1/\pi^2$ . Indeed, for each  $j$  consider the measure  $\mu_j$  defined by

$$\mu_j(B) := |\mathcal{H}_j \cap Q|_j^{-1} |\mathcal{H}_j \cap B|_j \quad \text{for all Borel } B \subset Q,$$

For  $\varphi \in C_{\#}^{\infty}$ ,  $\int_Q \varphi = 0$ ,  $\varphi \neq 0$ , we write

$$\varphi(x) = \sum_{l \in \mathbb{Z}^d \setminus \{0\}} c_l \exp(2\pi i l \cdot x), \quad x \in Q, \quad c_l \in \mathbb{C}, \quad l \in \mathbb{Z}^d,$$

and notice that for all  $j$  one has

$$\begin{aligned} & \left( \int_Q |\varphi|^2 d\mu_j \right)^{-1} \int_Q |i\varphi\varkappa + \nabla\varphi|^2 d\mu_j \\ &= \left( \sum_{l, m \in \mathbb{Z}^d \setminus \{0\}} \alpha_{lm} c_l \overline{c_m} \right)^{-1} \left( \sum_{l, m \in \mathbb{Z}^d \setminus \{0\}} \alpha_{lm} c_l \overline{c_m} (\varkappa + 2\pi l) \cdot (\varkappa + 2\pi m) \right) \geq \pi^2, \end{aligned}$$

where

$$\alpha_{lm} := \int_Q \exp(2\pi i(l - m) \cdot x) d\mu_j(x) = \begin{cases} 1, & l = m, \quad x_l \text{ is parallel to } \text{supp}(\mu_j), \\ 0 & \text{otherwise.} \end{cases}$$

For each  $\varkappa \in [-\pi, \pi)^d$ , let  $J = J(\varkappa)$  be the index such that

$$\left( \int_Q |\varphi|^2 d\mu_J \right)^{-1} \int_Q |i\varphi\varkappa + \nabla\varphi|^2 d\mu_J = \min_j \left( \int_Q |\varphi|^2 d\mu_j \right)^{-1} \int_Q |i\varphi\varkappa + \nabla\varphi|^2 d\mu_j.$$

Then

$$\begin{aligned} & \left( \int_Q |\varphi|^2 d\mu \right)^{-1} \int_Q |i\varphi\varkappa + \nabla\varphi|^2 d\mu = \left( \sum_j \int_Q |\varphi|^2 d\mu_j \right)^{-1} \sum_j \int_Q |i\varphi\varkappa + \nabla\varphi|^2 d\mu_j \\ & \geq \left( \sum_j \int_Q |\varphi|^2 d\mu_j \right)^{-1} \sum_j \left\{ \left( \int_Q |\varphi|^2 d\mu_j \right)^{-1} \int_Q |i\varphi\varkappa + \nabla\varphi|^2 d\mu_j \int_Q |\varphi|^2 d\mu_j \right\} \\ & = \left( \int_Q |\varphi|^2 d\mu_J \right)^{-1} \int_Q |i\varphi\varkappa + \nabla\varphi|^2 d\mu_J \geq \pi^2. \end{aligned}$$

Taking the infimum with respect to all  $\varphi \in C_{\#}^{\infty}$ ,  $\int_Q \varphi = 0$ ,  $\varphi \neq 0$ , and then the infimum with respect to  $\varkappa \in [-\pi, \pi)^d$  implies that one can take  $C_P(\mu) = 1/\pi$ , as claimed.

## 6 Estimate for the “remainder” $\varepsilon^2 R_\theta^\varepsilon$

**Theorem 6.1.** *Suppose that  $\theta \neq 0$ ,  $\varepsilon > 0$ . For the solution  $R_\theta^\varepsilon$  to the problem (4.18) the following estimates hold with  $C > 0$ :*

$$\left\| R_\theta^\varepsilon - \int_Q R_\theta^\varepsilon \right\|_{L^2(Q)} \leq C \|F\|_{L^2(Q)}, \quad \left| \int_Q R_\theta^\varepsilon \right| \leq C \varepsilon^{-1} \|F\|_{L^2(Q)}. \quad (6.20)$$

*Proof.* Consider a sequence of functions  $\varphi_n \in C_\#^\infty$  that converges in  $L^2(Q)$  to  $R_\theta^\varepsilon$ , such that

$$\nabla(e_{\varepsilon\theta}\varphi_n) \xrightarrow{[L^2(Q)]^d} \nabla(e_{\varepsilon\theta}R_\theta^\varepsilon),$$

and, equivalently,

$$\nabla \left[ e_{\varepsilon\theta} \left( \varphi_n - \int_Q R_\theta^\varepsilon \right) \right] \xrightarrow{[L^2(Q)]^d} \nabla \left[ e_{\varepsilon\theta} \left( R_\theta^\varepsilon - \int_Q R_\theta^\varepsilon \right) \right].$$

It follows from (4.18) that

$$\int_Q A \nabla(e_{\varepsilon\theta}R_\theta^\varepsilon) \cdot \overline{\nabla(e_{\varepsilon\theta}\varphi_n)} + \varepsilon^2 \int_Q R_\theta^\varepsilon \overline{\int_Q \varphi_n} = \langle H_\theta^\varepsilon, 1 \rangle \int_Q R_\theta^\varepsilon + \left\langle H_\theta^\varepsilon, \varphi_n - \int_Q R_\theta^\varepsilon \right\rangle.$$

Furthermore, using the fact that  $\langle H_\theta^\varepsilon, 1 \rangle = 0$  and the formula

$$\nabla \varphi_n = \overline{e_{\varepsilon\theta}} \left\{ \nabla \left[ e_{\varepsilon\theta} \left( \varphi_n - \int_Q R_\theta^\varepsilon \right) \right] - \left( \varphi_n - \int_Q R_\theta^\varepsilon \right) \nabla e_{\varepsilon\theta} \right\},$$

where all the gradients are understood in the classical sense, we obtain

$$\begin{aligned} \int_Q A \nabla(e_{\varepsilon\theta}R_\theta^\varepsilon) \cdot \overline{\nabla(e_{\varepsilon\theta}\varphi_n)} + \varepsilon^2 \int_Q R_\theta^\varepsilon \overline{\int_Q \varphi_n} &= \int_Q (F + i\theta \cdot A \nabla(iN_j \theta_j) c_\theta \\ &\quad - i\varepsilon N_j \theta_j \theta \cdot A \theta c_\theta - \theta \cdot A \theta c_\theta - c_\theta) \overline{\left( \varphi_n - \int_Q R_\theta^\varepsilon \right)} \\ &\quad + c_\theta \int_Q e_{\varepsilon\theta} N_j \theta_j A \theta \cdot \overline{\left\{ \nabla \left[ e_{\varepsilon\theta} \left( \varphi_n - \int_Q R_\theta^\varepsilon \right) \right] - \left( \varphi_n - \int_Q R_\theta^\varepsilon \right) \nabla e_{\varepsilon\theta} \right\}}. \end{aligned} \quad (6.21)$$

Passing to the limit as  $n \rightarrow \infty$  yields

$$\begin{aligned} \int_Q A \nabla(e_{\varepsilon\theta}R_\theta^\varepsilon) \cdot \overline{\nabla(e_{\varepsilon\theta}R_\theta^\varepsilon)} + \varepsilon^2 \left| \int_Q R_\theta^\varepsilon \right|^2 &= \int_Q \left[ F - c_\theta \left( \theta \cdot A \nabla(N_j \theta_j) + (i\varepsilon N_j \theta_j + 1) \theta \cdot A \theta \right. \right. \\ &\quad \left. \left. + e_{\varepsilon\theta} N_j \theta_j \theta \cdot \overline{A \nabla e_{\varepsilon\theta}} + 1 \right) \right] \overline{\left( R_\theta^\varepsilon - \int_Q R_\theta^\varepsilon \right)} + c_\theta \int_Q e_{\varepsilon\theta} N_j \theta_j \theta \cdot A \nabla \left\{ e_{\varepsilon\theta} \left( R_\theta^\varepsilon - \int_Q R_\theta^\varepsilon \right) \right\}. \end{aligned} \quad (6.22)$$

Consider the solution  $\Phi_\theta^\varepsilon \in H_\#^1$  to the problem

$$-\overline{e_{\varepsilon\theta}} \nabla \cdot A \nabla(e_{\varepsilon\theta}\Phi_\theta^\varepsilon) + \varepsilon^2 \int_Q \Phi_\theta^\varepsilon = -\overline{e_{\varepsilon\theta}} \nabla \cdot (e_{\varepsilon\theta} N_j \theta_j A \theta) c_\theta, \quad (6.23)$$

so that for the last term in (6.22) we obtain

$$c_\theta \int_Q e_{\varepsilon\theta} N_j \theta_j \theta \cdot A \nabla \left\{ e_{\varepsilon\theta} \left( R_\theta^\varepsilon - \int_Q R_\theta^\varepsilon \right) \right\} = \int_Q A \nabla(e_{\varepsilon\theta}\Phi_\theta^\varepsilon) \cdot \overline{\nabla \left\{ e_{\varepsilon\theta} \left( R_\theta^\varepsilon - \int_Q R_\theta^\varepsilon \right) \right\}}. \quad (6.24)$$

In what follows we use the uniform estimate

$$\|\sqrt{A}\nabla(e_{\varepsilon\theta}\Phi_\theta^\varepsilon)\|_{[L^2(Q)]^d} \leq C\|F\|_{L^2(Q)}, \quad (6.25)$$

which is obtained by using  $\Phi_\theta^\varepsilon$  as a test function in the integral formulation of (6.23).

We would like to rewrite the expression on the right-hand side of (6.24) using  $\Phi_\theta^\varepsilon$  as a test function in the integral identity for (4.18). Recall that the gradient of an arbitrary function in  $H_\#^1$ , for a general measure  $\mu$ , is not defined in a unique way. However, for the solution  $\Phi_\theta^\varepsilon$  to (6.23) there exists a natural choice of the gradient  $\nabla\Phi_\theta^\varepsilon$ , dictated by (6.23). Indeed, consider sequences  $\varphi_n, \psi_n \in C_\#^\infty$  converging to  $\Phi_\theta^\varepsilon$  in  $L^2(Q)$  so that

$$\nabla(e_{\varepsilon\theta}\varphi_n) \xrightarrow{[L^2(Q)]^d} \nabla(e_{\varepsilon\theta}\Phi_\theta^\varepsilon), \quad \nabla(e_{\varepsilon\theta}\psi_n) \xrightarrow{[L^2(Q)]^d} \nabla(e_{\varepsilon\theta}\Phi_\theta^\varepsilon).$$

Clearly, the difference  $\nabla(e_{\varepsilon\theta}\varphi_n) - \nabla(e_{\varepsilon\theta}\psi_n)$  converges to zero, and hence so does  $\nabla\varphi_n - \nabla\psi_n$ . In what follows we denote by  $\nabla\Phi_\theta^\varepsilon$  the common  $L^2$ -limit of gradients  $\nabla\varphi_n$  for sequences  $\varphi_n \in C_\#^\infty$  with the above properties. Passing to the limit, as  $n \rightarrow \infty$ , in the identity  $\nabla\varphi_n = \overline{e_{\varepsilon\theta}}(\nabla(e_{\varepsilon\theta}\varphi_n) - i\varepsilon\varphi_n\theta)$ , we obtain the formula

$$\nabla\Phi_\theta^\varepsilon = \overline{e_{\varepsilon\theta}}(\nabla(e_{\varepsilon\theta}\Phi_\theta^\varepsilon) - i\varepsilon\Phi_\theta^\varepsilon\theta). \quad (6.26)$$

The unique choice of  $\nabla\Phi_\theta^\varepsilon$ , as above, allows us to write

$$\int_Q A\nabla(e_{\varepsilon\theta}R_\theta^\varepsilon) \cdot \overline{\nabla(e_{\varepsilon\theta}\Phi_\theta^\varepsilon)} + \varepsilon^2 \int_Q R_\theta^\varepsilon \overline{\int_Q \Phi_\theta^\varepsilon} = \langle H_\theta^\varepsilon, \Phi_\theta^\varepsilon \rangle \equiv \left\langle H_\theta^\varepsilon, \Phi_\theta^\varepsilon - \int_Q \Phi_\theta^\varepsilon \right\rangle,$$

so that

$$\begin{aligned} \int_Q A\nabla(e_{\varepsilon\theta}\Phi_\theta^\varepsilon) \cdot \overline{\nabla\left\{e_{\varepsilon\theta}\left(R_\theta^\varepsilon - \int_Q R_\theta^\varepsilon\right)\right\}} &= \overline{\left\langle H_\theta^\varepsilon, \Phi_\theta^\varepsilon - \int_Q \Phi_\theta^\varepsilon \right\rangle} \\ &- \overline{\int_Q R_\theta^\varepsilon \left(\int_Q A\nabla(e_{\varepsilon\theta}\Phi_\theta^\varepsilon) \cdot \overline{\nabla e_{\varepsilon\theta}} + \varepsilon^2 \int_Q \Phi_\theta^\varepsilon\right)} = \left\langle H_\theta^\varepsilon, \Phi_\theta^\varepsilon - \int_Q \Phi_\theta^\varepsilon \right\rangle, \end{aligned} \quad (6.27)$$

where the values of the functional  $H_\theta^\varepsilon$  are chosen accordingly. In the last equality in (6.27) we use the fact that

$$\int_Q A\nabla(e_{\varepsilon\theta}\Phi_\theta^\varepsilon) \cdot \overline{\nabla e_{\varepsilon\theta}} + \varepsilon^2 \int_Q \Phi_\theta^\varepsilon = 0,$$

by setting the unity as a test function in the integral formulation of (6.23) and recalling that (cf. (4.11))

$$\int_Q AN_j = 0, \quad j = 1, 2, \dots, d.$$

Combining (6.22), (6.24) and (6.27) yields

$$\begin{aligned} \int_Q A\nabla(e_{\varepsilon\theta}R_\theta^\varepsilon) \cdot \overline{\nabla(e_{\varepsilon\theta}R_\theta^\varepsilon)} + \varepsilon^2 \left| \int_Q R_\theta^\varepsilon \right|^2 &= \int_Q \left[ F - c_\theta \left( \theta \cdot A\nabla(N_j\theta_j) + (i\varepsilon N_j\theta_j + 1)\theta \cdot A\theta \right. \right. \\ &\left. \left. + e_{\varepsilon\theta}N_j\theta_j \cdot \overline{A\nabla e_{\varepsilon\theta}} + 1 \right) \right] \overline{\left( R_\theta^\varepsilon - \int_Q R_\theta^\varepsilon \right)} + \left\langle H_\theta^\varepsilon, \Phi_\theta^\varepsilon - \int_Q \Phi_\theta^\varepsilon \right\rangle. \end{aligned} \quad (6.28)$$

**Lemma 6.2.** *The last term on the right-hand side of (6.28) is bounded, uniformly in  $\varepsilon, \theta$ :*

$$\left| \left\langle H_\theta^\varepsilon, \Phi_\theta^\varepsilon - \int_Q \Phi_\theta^\varepsilon \right\rangle \right| \leq C\|F\|_{L^2(Q)}, \quad C > 0.$$

*Proof.* Notice that

$$\begin{aligned} \left\langle H_\theta^\varepsilon, \Phi_\theta^\varepsilon - \int_Q \Phi_\theta^\varepsilon \right\rangle &= \int_Q \left( F + i\nabla \cdot (iN_j \theta_j A \theta) c_\theta + i\theta \cdot A \nabla (iN_j \theta_j) c_\theta \right. \\ &\quad \left. - i\varepsilon N_j \theta_j \theta \cdot A \theta c_\theta - \theta \cdot A \theta c_\theta - c_\theta \right) \left( \Phi_\theta^\varepsilon - \int_Q \Phi_\theta^\varepsilon \right) + c_\theta \int_Q N_j \theta_j A \theta \cdot \nabla \Phi_\theta^\varepsilon, \end{aligned} \quad (6.29)$$

where the second term is re-written using (6.26):

$$\int_Q N_j \theta_j A \theta \cdot \nabla \Phi_\theta^\varepsilon = \int_Q \overline{e_{\varepsilon\theta}} N_j \theta_j A \theta \cdot \nabla (e_{\varepsilon\theta} \Phi_\theta^\varepsilon) - i\varepsilon \int_Q \overline{e_{\varepsilon\theta}} N_j \theta_j A \theta \cdot \theta \left( \Phi_\theta^\varepsilon - \int_Q \Phi_\theta^\varepsilon \right).$$

Applying the Hölder inequality to both terms on the right-hand side of (6.29), using the Poincaré inequality (4.14) for  $\Phi_\theta^\varepsilon$ , and taking into account the bound (6.25) yields the required estimate.  $\square$

Combining the above lemma, the Poincaré inequality (4.14) for  $R_\theta^\varepsilon$  and Hölder inequality for the first term on the right-hand side of (6.28), we obtain the uniform bound

$$\|\sqrt{A} \nabla (e_{\varepsilon\theta} R_\theta^\varepsilon)\|_{[L^2(Q)]^d} \leq C \|F\|_{L^2(Q)}. \quad (6.30)$$

Finally, the bound (6.30) combined with (4.14) implies the first estimate in (6.20), whereas the same bound and equation (6.28) implies the second estimate in (6.20). This completes the proof of the theorem.  $\square$

Note that in the case  $\theta = 0$  the equality (6.22) takes the form

$$\int_Q A \nabla R_0^\varepsilon \cdot \overline{\nabla R_0^\varepsilon} + \varepsilon^2 \left| \int_Q R_0^\varepsilon \right|^2 = \int_Q F \overline{R_0^\varepsilon}, \quad (6.31)$$

and taking into account (4.14) with  $\varkappa = 0$ , we obtain

$$\|\nabla R_0^\varepsilon\|_{[L^2(Q)]^d} \leq C \|F\|_{L^2(Q)}.$$

The last estimate implies the first bound in (6.20) by (4.14) with  $\varkappa = 0$  and the second bound in (6.20) by using (6.31) once again.

**Corollary 6.3.** *The following estimate holds uniformly in  $\varepsilon > 0$ ,  $\theta \in \varepsilon^{-1}Q'$ ,  $F \in L^2(Q)$ :*

$$\|U_\theta^\varepsilon - c_\theta\|_{L^2(Q)} \leq C\varepsilon \|F\|_{L^2(Q)}.$$

## 7 Conclusion of the convergence estimate (4.15)

Here we estimate the error incurred by using the approximation  $U_\theta^\varepsilon$  in (3.10).

**Proposition 7.1.** *The difference  $z_\theta^\varepsilon := u_\theta^\varepsilon - U_\theta^\varepsilon$  satisfies the estimate*

$$\|z_\theta^\varepsilon\|_{L^2(Q)} \leq C\varepsilon \|F\|_{L^2(Q)}, \quad C > 0, \quad \forall \varepsilon > 0, \theta \in \varepsilon^{-1}Q', F \in L^2(Q).$$

*Proof.* It follows from (3.10), (4.17), (4.16), (4.11), (4.18), by a direct calculation, that

$$-\varepsilon^{-2} \overline{e_{\varepsilon\theta}} \nabla \cdot A \nabla (e_{\varepsilon\theta} z_\theta^\varepsilon) + z_\theta^\varepsilon = -i\varepsilon N_j \theta_j c_\theta - \varepsilon^2 \left( R_\theta^\varepsilon - \int_Q R_\theta^\varepsilon \right). \quad (7.32)$$

In particular, using  $z_\theta^\varepsilon$  as a test function in (7.32), we obtain

$$\varepsilon^{-2} \int A \nabla (e_{\varepsilon\theta} z_\theta^\varepsilon) \cdot \overline{\nabla e_{\varepsilon\theta} z_\theta^\varepsilon} + \int_Q |z_\theta^\varepsilon|^2 = -i\varepsilon c_\theta \theta_j \int_Q N_j \overline{z_\theta^\varepsilon} - \varepsilon^2 \int_Q \left( R_\theta^\varepsilon - \int_Q R_\theta^\varepsilon \right) \overline{z_\theta^\varepsilon},$$

and hence

$$\begin{aligned} \|z_\theta^\varepsilon\|_{L^2(Q)}^2 &\leq \varepsilon |c_\theta| |\theta| \|N\|_{[L^2(Q)]^d} \|z_\theta^\varepsilon\|_{L^2(Q)} + \varepsilon^2 C_P \|\nabla(e_{\varepsilon\theta} R_\theta^\varepsilon)\|_{[L^2(Q)]^d} \|z_\theta^\varepsilon\|_{L^2(Q)} \\ &\leq \varepsilon \left( |c_\theta| |\theta| \|N\|_{[L^2(Q)]^d} + \varepsilon C \|\sqrt{A} \nabla(e_{\varepsilon\theta} R_\theta^\varepsilon)\|_{[L^2(Q)]^d} \right) \|z_\theta^\varepsilon\|_{L^2(Q)}, \end{aligned}$$

where we use Proposition 4.14 once again and the fact that  $A$  is uniformly positive. The claim follows, by virtue of the formula (4.16) and the estimate (6.30).  $\square$

Combining Corollary 6.3 and Proposition 7.1 concludes the proof of Theorem 4.2.

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